Accounting for *primitive terms* in mathematics[1]

D.F.M. Strauss
Faculty of Humanities
University of the Free State
BLOEMFONTEIN
E-mail: dfms@cknet.co.za

> Practising mathematicians, consciously or not, subscribe to some philosophy of mathematics (if unstudied, it is usually inconsistent). (Monk, 1970:707.)

Abstract

Accounting for *primitive terms* in mathematics

The philosophical problem of unity and diversity entails a challenge to the rationalist aim to define everything. Definitions of this kind surface in various academic disciplines in formulations like uniqueness, irreducibility, and what has acquired the designation "primitive terms". Not even the most "exact" disciplines, such as mathematics, can avoid the implications entailed in giving an account of such primitive terms. A mere look at the historical development of mathematics highlights the fact that alternative perspectives prevailed – from the arithmeticism of Pythagoreanism, the eventual geometrisation of mathematics after the discovery of incommensurability up to the revival of arithmeticism in the mathematics of Cauchy, Weierstrass, Dedekind and Cantor (with the later orientation of Frege, who completed the circle by returning to the view that mathematics essentially is geometry). An assessment of logicism and axiomatic formalism is followed by looking at the primitive meaning of wholeness (and the whole-parts relation). With reference to the views of Hilbert, Weyl and Bernays the article concludes by suggesting that in

1 I want to thank two mathematicians – Proff. B. de la Rosa and H. Bargenda – for many valuable suggestions made during thorough discussions of the text of an earlier version of this article.

opposition to arithmeticism and geometricism an alternative option ought to be pursued – one in which both the uniqueness and mutual coherence between the aspects of number and space are acknowledged.

Opsomming

Die verantwoording van *primitiewe terme* in die wiskunde

Die wysgerige probleem van eenheid en verskeidenheid bevat 'n uitdaging vir die rasionalistiese strewe om alles te definieer. Hierdie tipe definisies kom in verskillende akademiese dissiplines na vore in formulerings soos uniekheid, onherleibaarheid, en wat bekend staan as "primitiewe terme". Selfs nie die mees "eksakte" dissiplines, soos die wiskunde, kan daarin slaag om te ontkom aan die implikasies wat opgesluit lê in die rekenskapgewing of verantwoording van sulke primitiewe terme nie. Deur te let op die historiese ontwikkeling van die wiskunde word die feit belig dat alternatiewe perspektiewe deurlopend bestaan het – vanaf die aritmetisisme van die Pytagoreërs, die eventuele geometrisering daarvan ná die ontdekking van "inkommensurabiliteit", tot by die herlewing van die aritmetisisme in the wiskunde van Cauchy, Weierstrass, Dedekind en Cantor (met die latere oriëntasie van Frege teen die einde van sy lewe toe hy die sirkel voltooi het met sy oortuiging dat die wiskunde wesenlik geometrie is). Met verwysing na die sienings van Hilbert, Weyl en Bernays word die artikel afgesluit deur die voorstel dat daar teenoor die erfenis van aritmetisisme en geometrisering 'n alternatiewe opsie ontgin behoort te word – een waarin sowel die uniekheid as die onverbreeklike wederkerige samehang tussen die getalsaspek en die ruimte-aspek erken word.

1. Introduction

Although every single academic discipline employs *concepts,* an explicit account of the *nature* of concept-formation is almost never encountered – in general the *concept of a concept* is absent. Normally key (or supposedly *basic*) concepts are *defined* straightaway. Every definition, however, has to use certain *terms* and this fact gives rise to questions such as:

- Is it possible to define every one of the used terms?
- Can this process be continued indefinitely or do we somewhere along the line encounter terms that cannot be defined any more?

That an endless sequence of successive *definitions* merely results in the logical fallacy known as a *regressus in infinitum* has been recognised by various disciplines.[2] Ultimately every scholarly discipline therefore has to come to terms with its *primitive terms*, i.e. those terms that cannot be defined any further, but that constitute the inevitable building blocks for every definition. The rationalist ideal that everything should be *defined,* in this respect runs into the *self-insufficiency* of definition and concept formation, for ultimately the latter rests upon *terms* that are *not defined and* cannot be defined.

How does one *know* these indefinable (primitive) terms? This is an epistemological issue that is rooted in *philosophical assumptions* about the world in which we live – and therefore involve *ontological commitments*.

Surely there are multiple *primitive terms* – implying that this multiplicity of primitive terms brings to expression a fundamental *diversity* within reality. For this reason one should recognise the intimate connection between the problem of *primitive terms* and the problem of *unity and diversity*. The reverse side of this coin is also known as the problem of irreducibility, and is sometimes designated as complementarity. Hoyningen-Huene, for example, explicitly writes about *irreducibility* in the context of *complementarity*:

> But this property is just *identical with the epistemological non-reducibility* of these features. In other words: in order to establish that in a certain situation complementarity prevails, it has to be shown that the features involved are irreducible to each other. (Hoyningen-Huene, 1991:67).

Weingartner (1991:124) thus refers to *primitive terms*:

> Term (concept, idea) *t* is scientifically analysable if it is reducible to primitive terms. *t* is reducible to primitive terms if *t* is itself a primitive term or it can be traced back to primitive terms by a chain of definitions.

2. Historical contours

Within the history of philosophy this underlying issue of *unity* and *diversity* could be designated as the problem of the *coherence of irreducibiles*.

2 Salmon, for example, refers to *primitive terms* in “pure mathematics” (Salmon, 2001:32).

The basic orientation of Greek mathematics in its initial phase is succinctly captured by the Pythagorean statement: "everything is number". The discovery of incommensurability by Hippasos of Metapont (about 450 B.C. – cf. Von Fritz, 1945) generated an emphasis on a new approach that attempted to consider our intuition of space as being *more basic* than that of number. Because intuition can grasp continuity at once "Greek mathematics and philosophy were inclined to consider continuity to be the simpler concept" (Fraenkel *et al*., 1973:213). During the later part of the 19th century mathematics once again reverted to an arithmeticistic perspective – a process initiated by Bolzano and carried through by Weierstrass, Dedekind and Cantor.

What is amazing about this course of events is that what seemed to have been solved twice still turned out to form the centre of the problems involved, that is the discovery of irrationality ("incommensurability") and the difficulties surrounding the *theory of functions* in Germany and France. Fraenkel *et al*. (1973:212-213) remark in this regard:

> … although the arguments have changed, the gap between discrete and continuous is again the weak spot – an eternal point of least resistance and at the same time of overwhelming scientific importance in mathematics, philosophy, and even physics.

It is indeed amazing that the entire history of mathematics only explored the following two options: either reduce space to number or reduce number to space. It should be kept in mind that Frege eventually showed an affinity with the latter position. Dummett refers to Frege's own "eventual expedient of reducing arithmetic to geometry" (Dummett, 1995:319). With regard to a preference for geometrical intuitions Wang (1988:203) writes: "An alternative course would be to consider our geometrical intuitions, as Plato and Bernays (and, I understand, also Frege in his later years) apparently preferred." The intellectual development of Frege is indeed most striking in this context. In 1884 he published a work on the foundations of arithmetic. After his first volume on the *basic laws of arithmetic* was published in 1893, Russell's discovery (in 1900) of the antinomous character of Cantor's set theory[3] for some time

3 Russell considered the set C with sets as elements, namely all those sets A that do not contain themselves as an element. It turned out that if C is an element of itself it must conform to the condition for being an element, which stipulates that

delayed the publication of the second volume in 1903 – where he had to concede in the first sentence of the Appendix that one of the corner stones of his approach had been shaken. Close to the end of his life, in 1924/1925, Frege not only reverted to a *geometrical source of knowledge*, but also explicitly rejected his initial *logicist* position. In a sense he completed the circle – analogous to what happened in Greek mathematics after the discovery of irrational numbers. In the case of Greek mathematics this discovery prompted the geometrisation of their mathematics, and in the case of Frege the discovery of the untenability of his *Grundlagen* also inspired him to hold that mathematics as a whole actually is *geometry*:

> An *a priori* mode of cognition should therefore be involved in this respect. This cognition does, however, not have to flow from purely logical principles, as I originally assumed. There is the further possibility that it has a geometrical source.
>
> The more I have thought the matter over, the more convinced I have become that arithmetic and geometry have developed on the same basis – a geometrical one in fact – so that mathematics in its entirety is really geometry (Frege, 1979:277).

The obvious *third* option was never examined: accept both the uniqueness (i.e., *irreducibility*) of number and space *and* their mutual interconnectedness. In this article the focus will be on diverging mathematical positions caused by alternative accounts of the basic ("primitive") elements shaping the direction which mathematics as a discipline took.

3. Probing some foundational designs

Although Euclid already employed the axiomatic method it received its modern rigour from Hilbert's work on the foundations of geometry (*Grundlagen der Geometrie*, 1922 [1899]). In this work Hilbert abstracts from the contents of his axioms, based upon three *undefined* terms: *point, lies on,* and *line*.

3.1 The impasse of logicism

Frege, Dedekind and Russell advanced a *logicistic approach* to the foundations of mathematics. Dedekind started from an actual infinity of "objects" within my *Gedankenwelt* (translated by Rucker with the

it cannot be an element of itself. Conversely, if C is not an element of itself, it indeed obeys the condition for being an element of itself!

descriptive term *mindscape* – Rucker, 1982:47). Russell defines *number* with the aid of his supposedly *purely logical* concept of *class*. The logical concept, he claims, enables the reduction of mathematics to logic. For example, the number "2" is "defined" in the following way:

> 1 + 1 is the number of a class *w* which is the logical sum of two classes *u* and *v* which have no common terms and have each only one term. The chief point to be observed is, that logical addition of numbers is the fundamental notion, while arithmetical addition of numbers is wholly subsequent (Russell, 1956:119).

The irony, however, is that Russell already had to use the *meaning of number* in order to distinguish between different ("logical") classes. After all, he speaks about the sum of "two" classes where each of them contains "one" element. This presupposes an insight into the *numerical meaning* of the numbers "1" and "2"! Consequently, the number "2," which had to appear as the *result* of "logical addition," is *presupposed* by it. In his discussion of *number and the concept of class* Cassirer displays a clear understanding of this circularity (cf. Casirer, 1953:44 ff.).[4]

Although Dedekind asserts that the *idea of infinity* should form part of the *logical foundation* of mathematics, it soon turned out that the meaning of infinity *precedes* logic. Hilbert points out that in contrast to the early attempts of Frege and Dedekind he is convinced that as a precondition for the possibility of scientific knowledge certain intuitive representations and insights are indispensible and that logic alone is not sufficient.[5] Fraenkel *et al.* (1952:182) also affirm: "It seems, then, that the only really serious drawback in the Frege-Russell thesis is the doubtful status of InfAx,[6] according to the interpretation intended by them". Myhill (1952:182) mentions the fact that the axioms of *Principia* do not determine how many individuals there are: "... the axiom of infinity, which is needed as a hypothesis

---

4 Singh (1985:76) also points out that Russell's attempt makes him a victim of the "vicious circle principle".

5 "Im Gegensatz zu den früheren Bestrebungen von Frege und Dedekind erlangen wir die Überzeugung, daß als Vorbedingung für die Möglichkeit wissenschaftlicher Erkenntnis gewisse anschauliche Vorstellungen und Einsichten unentbehrlich sind und die Logik allein nicht ausreicht" (Hilbert, 1925:190).

6 InfAx = *Axiom of Infinity*.

for the development of mathematics in that system, is neither provable nor refutable therein, *i.e.*, is undecidable".

Every attempt to derive the meaning of number from the meaning of analysis (or logic) is indeed faced with a *vicious circle*. Cassirer is also quite explicit in this regard. He claims that a critical analysis of knowledge, in order to side-step a *regressus in infinitum*, has to accept certain *basic functions* which are not capable of being "deduced" and which are not in need of a deduction.[7] David Hilbert also points at this "catch 22" entailed in the logicist attempt to deduce the meaning of number from that of the logical-analytical mode. In his *Gesammelte Abhandlungen* Hilbert (1970:199) writes:

> Only when we analyze attentively do we realize that in presenting the laws of logic we already had to employ certain arithmetical basic concepts, for example the concept of a set and partially also the concept of number, particularly as cardinal number [Anzahl]. Here we end up in a vicious circle and in order to avoid paradoxes it is necessary to come to a partially simultaneous development of the laws of logic and arithmetic.

3.2 Contradiction and the meaning of analysis

The initial stage of mathematical set theory, as developed by Georg Cantor (between 1874 and 1899), got stuck in contradictions. By 1895 Cantor himself discovered that his set theory contains anomalies. Cantor proved e.g. the proposition that for every set *A* of ordinal numbers an ordinal number exists which is greater than every ordinal number contained in the set. Consider, however, the set *W* of *all* ordinal numbers. Since this set is a set of all ordinal numbers, the foregoing proposition implies that an ordinal number exists which is greater than every ordinal number contained in *W* – but this is contradictory, since the set *W* supposedly already contains *all* ordinal numbers. A similar contradiction holds with regard to Cantor's cardinal numbers (cf. Meschkowski, 1967:144-145 and Singh, 1985:73).

The ensuing *axiomatisation* of set theory, for example that of Zermelo-Fraenkel, proceeds on the basis of (i) *first-order predicate*

7 "Denn die kritische Analyse der Erkenntnis wird, wenn man nicht einen regressus in infinitum annehmen will, immer bei gewissen Urfunktionen Halt machen müssen, die einer eigentlichen 'Ableitung' weder fähig noch bedurftig sind" (Cassirer, 1957:73).

calculus[8] and (ii) it introduces as undefined term the specific set-theoretical primitive binary predicate є which is called the *membership relation* (Fraenkel *et al*., 1973:23).[9] This approach follows a general pattern: An axiomatic theory (axiomatic theories of logic excluded)

> ... is constructed by adding to a certain *basic discipline* – usually some system of logic (with or without a set theory) but sometimes also a system of arithmetic – new terms and axioms, the *specific* undefined terms and axioms under consideration (Fraenkel *et al*., 1973:18).

The first-order predicate calculus assumed by Zermelo-Fraenkel (ZF) contains a set of *connectives* enabling the expression of *negation, conjunction, disjunction, conditional, biconditional* and the two *quantifiers: the universal quantifier* (*for all*) and *the existential quantifier* (*there exist*).

This means that the underlying logic provides ZF Set Theory with the following primitive symbols: *connectives*, *quantifiers* and, in addition, also *variables*.

These connectives testify to the fact that the analytical mode is *normed* in the sense that it makes possible *concept-formation* and *argumentation* which may or may not conform to normative logical principles (such as the principle of identity, (non-)contradiction, the excluded middle, and so on). *Negation* makes it possible to assess *illogical* thinking and *contradictions* – a statement and its negation cannot both be true in the same context (at the same time). *Negation* as a connective presupposes the logical principles of identity and (non-) contradiction – in what is analysable A is A and A is not non-A. Thus analysis presupposes *unity* and *multiplicity.* First of all this relates to *arithmetical phenomena.* Therefore the logical principle of identity and that of non-contradiction *analogically reflect* this basic arithmetical meaning of *unity and multiplicity*. Whatever is

---

8 Van Heijenoort (1967:285) remarks that "an axiomatization of set theory is usually embedded in a logical calculus, and it is Weyl's and Skolem's approach to the formulation of the axiom of separation that is generally adopted".

9 Note that within Zermelo-Fraenkel Set Theory (ZF) the terms "set" and "element" are synonymous (Fraenkel *et al*., 1973:24), implying that this theory will avoid the phrase "*x* is an element of *y*" (Fraenkel *et al*., 1973:23, note 2). Their terminology, in terms of the membership relation, is such that "$x \in y$" is read as "*x* is a member of *y*" or as "*x* belongs to *y*" (which is synonymous with "*x* is contained in *y*" / "*y* contains *x* (as a member)" (Fraenkel *et al*., 1973:23).

given as a discrete unity (as being distinct) is *identical* to itself (the basis of the principle of identity) and is *different* from whatever it is not (the basis of the principle of non-contradiction) (cf. Strauss, 1991). In themselves these two principles therefore (in a positive and negative way) reveal the *coherence* between two *irreducible* modal aspects of reality, namely the *logical* and the *numerical* modes.

We may note that *arithmetical addition* differs from a *logical synthesis* – something already understood by Immanuel Kant. Where he argues for the *synthetic* nature of mathematical judgments in his *Critique of Pure Reason* (CPR), he clearly realises that pure logical addition (a mere logical synthesis) cannot give rise to a *new number* (cf. CPR, 1787:15 where he considers the proposition that 7+5=12). In a different way Frege made the same point: abstraction directed towards concretely existing entities can only proceed to more abstract entities, but it can never result in the specification of any *number* as such. The logical addition of “ones” or “twos” cannot but end with the repeated identification of another number of the same kind: having identified a “two” and another “two” still provides us only with the “abstract” notion of “twoness”. Logical addition therefore cannot be equated with arithmetical addition.

Likewise we have to distinguish between the kinematical notion of uniform flow (constant movement) and the physical meaning of energy-operation (causing certain effects, i.e. change/variation) on the one hand, and the idea of logical *constants* and *variables* on the other.

In the context of the preceding remarks about the difference between logical addition and arithmetical addition it is worth noting that the presence of *variables* within first-order predicate calculus in yet another way also highlights the *original meaning* of number – but this time in coherence with the *kinematical* and the *physical* aspects. Physical changes presuppose *constancy*, keeping in mind that uniform motion (constancy) has a *kinematical* meaning, particularly surfacing in Einstein’s special theory of relativity where the velocity of light is postulated as being *constant* in a vacuum.

The conditional as connective analogically reflects the physical cause-effect relation – if *this* then one can conclude to *that*. Weyl (1966:32) touches upon the relation between the (physical) relationship between cause and effect on the one hand, and ground and conclusion on the other, without realising that in this respect we encounter a *basic concept* of logic – one that analogically reflects the coherence between the logical and the physical aspects of reality. The relation between logical grounds and conclusions reflects the original physical relation of cause and effect. He does realise that the logical relation may have its foundation in an applicable essential law, a "causal connection" or "an empirical regularity", but nonetheless claims that the sign → "expresses a *purely* logical conclusion (*Folge*)" – without realising that the inevitability of employing this term *Folge* demonstrates the undeniable analogical link between the logical and the physical aspects of reality.

In the footsteps of Plato and Galileo it became clear to modern physics and to Einstein that *changes* could only be detected on the basis of *constancy*, explaining the close connection of *constants* and *variables* also in logic.[10]

Furthermore, the mere fact that both these terms appear in the *plural* makes it plain that the practice in logic to employ the terms *constants* and *variables* at once also presupposes the original *quantitative meaning* of number.

3.3 Tacit assumptions of axiomatic set theory

The first four axioms of ZF do not guarantee the existence of a set (or an object) at all (cf. Fraenkel *et al*., 1973:39, note 2). These Axioms are (Fraenkel *et al*., 1973:27-35):

- The Axiom of Extensionality (if $x \subseteq y$ and $y \supseteq x$, then $x = y$);
- the Axiom of Pairing (for any two elements *a* and *b* there exists the set *y* which contains just *a* and *b* (i.e., *a* and *b* and no different member));
- the Axiom of Union / Sumset (for any set *a* there exists the set whose members are just the members of the members of *a*);
- the Axiom of Powerset (for any set *a* there exists the set whose members are just all the subsets of *a*).

In addition to these four Axioms

10 Weierstrass tried to eliminate the notion of variables through the postulation of a static infinite domain.

- the *Axiom of Subsets* and the *Axiom of Infinity* are introduced.[11] The Axiom of Infinity asserts explicitly that some set exists (cf. Fraenkel *et al*., 1973:39, note 2).[12]

In the next part of the discussion some of the tacit assumptions of axiomatic set theory will be highlighted.

- In the absence of a foundational *ontological* consideration of the interrelationships between the numerical, the spatial, the kinematical, the physical and the analytical aspects of reality an extremely fundamental *circulus vitiosus* is actually concealed by set theory and its logic.
- Set theory is seen as a purely *arithmetical* theory (as it was in fact intended by Cantor and most of his successors). Yet, in order to construct an *axiomatic foundation* for set theory, the aid of an underlying *logic* is required. Does this mean that logic in itself can provide a *sufficient* foundation for set theory (or mathematics)?
- Whereas Russell (1956:v) claims that logic and mathematics are *identical*, and even made an attempt to derive the number concept from the logical class concept, it has already been indicated in this article that this entire procedure begs the question. The claims of logicism are untenable because both the logical class concept as well as the Axiom of Infinity make an appeal to the *basic meaning of number*. In other words, the *meaning* of analysis (i.e the logical-analytical mode of reality) presupposes the *meaning* of number and therefore cannot serve as a foundation for it.

From the preceding analysis and argumentation it may therefore be concluded that the pretended foundation of set theory in logic attempts to side-step the crucial issue. In order to provide an axiomatic foundation for an analysis of the *meaning of number* an

---

11 The former (also known as the Axiom of Separation) states that for any set *a* and any condition *B*(*x*) on *x* there exists the set that contains just those members *x* of *a* which fulfil the condition *B*(*x*), while the latter (the *Axiom of Infinity*: InfAx) secures the existence of infinite sets by postulating them (cf. Fraenkel *et al*., 1973:46). Presently the discussion will return to the form of the InfAx.

12 For the purposes of the present discussion the Axiom Schema of Replacement/Substitution, the Axiom of Choice and the Axiom Schema of Foundation will be left aside.

underlying logic is required which in itself presupposes this basic meaning of number!

As already pointed out this view already follows from the inevitable presence of *quantifiers* and the presence of a *multiplicity* of (*constants* and) *variables* assumed in first-order predicate calculus. The intuition (basic awareness) of *multiplicity* is made possible by the unique quantitative meaning of the numerical aspect of reality – first of all accounted for in a systematic mathematical understanding of the *natural numbers* and in the fact that *succession* is also inherent to our comprehension of natural numbers (the best known mathematical "application" of this order of succession is found in [mathematical/complete] *induction*).

According to Freudenthal, Dedekind perhaps was the first mathematician (cf. Dedekind, 1887: par. 59, 80) to call the conclusion from *n* to *n + 1 complete induction* (*vollständige Induktion*). Neither Bernoulli nor Pascal is the founder of this principle. Its discovery must be credited to Francesco Maurolico (1494-1575) (cf. Freundenthal, 1940:17). In a mathematical context, where "bad induction" is supposed to be excluded (as Freundenthal remarks – 1940:37), no adjective is necessary to qualify the term *induction*. Already in 1922 Skolem had a solid understanding of these issues:

> Those engaged in doing set theory are normally convinced that the concept of an integer ought to be defined and that complete induction must be proved. Yet it is clear that one cannot define or provide an endless foundation; sooner or later one encounters what is indefinable and what cannot be proved. The only option left is to ensure that the first starting points are immediately clear, natural and beyond doubt. The concept of an integer and the inferences by induction meet this condition, but it is definitely not met by the set theoretic axioms such as those of Zermelo or similar ones. If one wishes to derive the former concepts from the latter, then the set theoretic concepts ought to be simpler and employing them then ought to be more certain than working with complete induction – but this contradict the real state of affairs totally (Skolem, 1979 [1929]:70).[13]

---

13 "Die Mengentheoretiker sind gewöhnlich der Ansicht, dass der Begriff der ganzen Zahl definiert werden soll, und die vollständige Induktion bewiesen werden soll. Es ist aber klar, dass man nicht ins Unendliche definieren oder begründen kann; früher oder später kommt man zu dem nicht weiter Definierbaren bzw. Beweisbaren. Es ist dann nur darum zu tun, dass die ersten Anfangsgründe etwas unmittelbar Klares, Natürliches und Unzweifelhaftes sind. Diese Bedingung ist für den Begriff der ganzen Zahl und die Induktionsschlüsse

4. The primacy of natural numbers and their succession (induction)

The intuition of one, another one, and so on, generates the most basic meaning of infinity – literally without an end, *endlessly*, *infinitely*. The awareness of a *multiplicity* is at least accompanied by an awareness of *succession*.[14]

Weyl states that the startingpoint of mathematics is the series of natural numbers, the law according to which the number 1 is brought forth from nothing and where every number in turn gives rise to its successor (Weyl, 1921:57). On the next page Weyl calls the "always another one" the original mathematical intuition (*die mathematische Urintuition*) – it is not possible and it is not required to provide a further foundation for this *Urintution* (basal intuition). Upon this basis Weyl categorically holds that from the intuitionistic standpoint *complete induction* secures mathematics from being an enormous tautology and impregnates its statements as synthetic (non-analytic).[15]

With reference to Weyl also Skolem accentuates the fact that the concept of an integer and of induction constitutes the logical content of Hilbert's metamathematics.[16] Number displays an *order of*

erfüllt, aber entschieden nicht erfüllt für mengentheoretische Axiome der Zermelo'schen Art oder ähnliches; sollte man die Zurückführung der ersteren Begriffe auf die letzteren anerkennen, so müssten die mengentheoretischen Begriffe einfacher sein und das Denken mit ihnen unzweifelhafter als die vollständige Induktion, aber das läuft dem wirklichen Sachverhalt gänzlich zuwider.

14 Kant realised that succession differs from causation: day succeeds night and night succeeds day, but neither day nor night causes night or day. It shall be argued below that whenever a multiplicity is grasped collectively (i.e., at once), an appeal is made to our intuition of space.

15 "Unabhängig aber davon, welchen Wert man dieser letzten Reduktion des mathematischen Denkens auf die Zweieinigkeit beimißt, erscheint vom intuitionistischen Standpunkt die vollständige Induktion als dasjenige, was die Mathematik davon bewahrt, eine ungeheure Tautologie zu sein, und prägt ihren Behauptungen einen synthetischen, nicht-analytischen Charakter auf" (Weyl, 1966:86).

16 "In der Tat basiert sich ja Hilbert sehr wesentlich auf dem Begriff der ganzen Zahl und der vollständigen Induktion in der Metamathematik, und diese stellt ja den logischen Inhalt seiner Theorie dar" (Skolem, 1929:89). What Skolem calls the "logical content" actually refers to the primitive meaning of number which is presupposed in logic – an insight also emphasised by Weyl (see the main text below and the next footnote).

succession that is embodied in the application of induction. It is therefore significant that De Morgan already in 1838 used the equivalent expression *successive induction* (cf. Freundenthal, 1940:36-37). Weyl does not stop to emphasise that mathematics in its entirety, even regarding the logical form in which it operates, is dependent upon the essence of *natural numbers*.[17]

5. Wholeness and totality – the irreducibility of the whole-parts relation

Although the terms *whole* and *totality* are closely related to the term *continuity* it seems difficult to *define* the meaning of *continuous extension* (as realised by Dantzig, 1947:167 – see also Strauss, 2002). Synonyms like *uninterrupted*, *connected*, *coherent*, and so on, simply repeat what is meant by continuity, instead of *defining* it. Yet, that a "continuous" whole allows for an *infinite number of divisions* was already discovered by the Greeks. Zeno's B Fr. 3 reads as follows:

> If things are a multiplicity, then it is necessary that their number must be identical to their actual multiplicity, neither more nor less. But if there are just as many as there are, then their number must be limited (finite). If things are a multiplicity, then necessarily they are infinite in number; for in that case between any two individual things there will always be other things and so on. Therefore, then, their number is infinite.

The assumption that *things are many* serves two opposite conclusions. Apparently the two *sides* of the (spatial) whole-parts relation provide the foundation for this argument. If the multiplicity of the first section refers to the many *parts* of the world as a *whole*, it stands to reason that taken together they constitute the *unity* of the world as a *whole* (and that their number would be *limited*). If, on the other hand, one starts with the *whole* and then tries to account for its *parts*, one must keep in mind that between any two of the many parts there will always be other, indicating an *infinity* of them. Fränkel explicitly uses the *whole-parts relation* to explain the meaning of this fragment (Fränkel, 1968:425 ff., 430).

If this interpretation is sound, then Zeno not only understood something of the whole-parts relation, but also, for the first time,

17 "Die Mathematik ist ganz und gar, sogar den logischen Formen nach, in denen sie sich bewegt, abhängig vom Wesen der natürlichen Zahl" (Weyl, 1921:70).

realised that the feature of infinite divisibility characterises spatial continuity. This perspective also supports the interpretation given to Zeno's B Fragment 1 by Hasse and Scholz (1928:10-13). This first fragment (which we inherited from Simplicius) states that if there is a multiplicity, then it must be simultaneously large and small; large up to infinity and small up to nothingness. Hasse and Scholz clarify this fragment by interpreting this view as follows:

> If it is permissible to conceptualize a line-stretch as an aggregate of infinitely many small line stretches, then there are two and only two possibilities. Every basic line segment either has a finite size (larger than zero), in which case the aggregate of line-stretches transcends every finite line-stretch; or the supposed line-stretches are zero-stretches in the strict sense of the word, in which case the composed line is also a zero-stretch, because the combination of zero-stretches can always only produce a zero-stretch, however large the number of zero-stretches used may be[18] (Hasse & Scholz, 1928:11).

Besides the fact that the two mentioned fragments of Zeno can be rendered perfectly intelligible by using the whole-parts relation, further support for this understanding may also be drawn from the account which Aristotle gave of Zeno's arguments (cf. *Metaph*. 233 a 13 ff. and 239 b 5 ff.; see Aristotle, 2001). One of the standard expositions of Zeno's argumentation against the reality of motion is completely dependent on the employment of the whole-parts relation with its implied trait of infinite divisibility. Guthrie (1980:91-92) explains this argument by saying that according to Zeno

> [m]otion is impossible because an object moving between any two points A and B must always cover half the distance before it gets to the end. But before covering half the distance it must cover half of the half, and so ad infinitum. Thus to traverse any distance at all it must cover an infinite number of points, which is impossible in any finite time.

For the larger part of 2000 years continuity (with its implied whole-parts relation), dominated the scene – both within the domains of philosophy and mathematics. Early modernity witnessed atomistic theories of nature, but only in the course of the nineteenth century it penetrated the mathematical treatment of continuity. Bolzano

18 It can be argued that Grünbaum, with the aid of non-denumerable sets, attempted to side-step the last remark in order to accomplish an assumed complete arithmetisation of the continuum.

explored an atomistic approach and it was brought to fruition by Weierstrass, Dedekind and Cantor.

The new assumption in the approach of Weierstrass holds that we have to define *limits* in terms of a *static domain* encompassing *all* real numbers. Boyer (1959:286) refers to this in the following explanation:

> In making the basis of the calculus more rigorously formal, Weierstrass also attacked the appeal to the intuition of continuous motion which is implied in Cauchy's expression – that a variable approaches a limit. Previous writers generally had defined a variable as a quantity or magnitude which is not constant; but since the time of Weierstrass it has been recognized that the ideas of variable and limit are not essentially phoronomic, but involve purely static considerations. Weierstrass interpreted a variable x simply as a letter designating any one of a collection of numerical values. A continuous variable was likewise defined in terms of static considerations: If for any value x_o of the set and for any sequence of positive numbers $d_1, d_2, ..., d_n$, however small, there are in the intervals $x_o - d_i, x_o + d_i$ others of the set, this is called continuous.

Weyl characterises the apparent success of this new aim to arithmetise mathematics as *atomistic* (Weyl, 1921:56, 72). According to him within a continuum it is certainly possible, through divisions, to generate partial *continua*, but it is not clever (*unvernünftig*) to assert that the total continuum is composed of the limits and these partial *continua*. "A true continuum after all coheres within itself and cannot be divided into separate pieces; it contradicts its essence."**19** Weyl mentions that he changed his own view by accepting the position taken by Brouwer in this regard. Nonetheless, one does not have to adhere to the intuitionistic understanding of (infinity and) continuity to realise that the *whole-parts relation* and the *totality-character of continuity* stands in the way of a complete arithmetisation of continuity (the outcome of an atomistic approach).

---

19 "Erscheint dies dem heutigen Mathematiker mit seiner atomistischen Denkgewöhnung anstößig, so war es in früheren Zeiten eine allen selbstverständliche Ansicht: innerhalb eines Kontinuums lassen sich wohl durch Grenzsetzung Teilkontinuen erzeugen; es ist aber unvernünftig, zu behaupten, daß das totale Kontinuum aus der Grenze und jenen Teilkontinuen zusammengesetzt sei. Ein wahrhaftes Kontinuum ist eben ein in sich Zusammenhängendes und kann nicht in getrennte Bruchstücke aufgeteilt werden; das widerstreitet seinem Wesen" (Weyl, 1921:73).

Paul Bernays, the co-worker of David Hilbert, senses the irreducibility of the spatial whole-parts relation (i.e. the *totality* feature of spatial continuity) with an astonishing lucidity. He asserts that the property of being a totality "undeniably belongs to the geometric idea of the continuum. And it is this characteristic which resists a complete arithmetization of the continuum".[20]

Compare his remark in a different context where he objects that the classical foundation of real numbers given by Cantor and Dedekind does not "manifest a complete arithmetization" (Bernays, 1976:187-188). To this he adds the remark: "It is in any case doubtful whether a complete arithmetization of the idea of the continuum could be justified. The idea of the continuum is any way originally a geometrical idea" (Bernays, 1976:188).

Something remarkable emerges from this situation. If the nature of totality (wholeness/the whole-parts relation) cannot be *arithmetised* how does one explain the entailed property of being *infinitely divisible*? Does this property imply that the (numerical) intuition of *succession* (literally without an end/endless/infinite) is an indispensible building block of continuity? Does it therefore mean that "number" is "built into" the nature of the (spatial) meaning of "continuity"?

6. Conclusion

As is evident from the history of mathematics and from what has been argued in connection with alternative trends within mathematics it can be concluded that this discipline (similarly to every other intellectual concern) constantly had to face the problem of uniqueness and coherence, particularly in its diverging responses to the *primitive terms* involved. Since axiomatic set theory assumes an underlying logic without critically accounting for the ontological assumptions operative in this logic, it actually *conceals* its own important ontological presuppositions. The on-going pendulum-shift between *arithmeticistic* and *geometricistic*[21] strategies within mathematics ought to be confronted with *a third option* not yet explored in the history of mathematics, that is acknowledging both

20 "Und es ist auch dieser Charakter, der einer vollkommenen Arithmetisierung des Kontinuums entgegensteht" (Bernays, 1976:74).

21 Ultimately the one-sidedness of monistic -*isms* like these demonstrate the negative effects of deifying some or other aspect of created reality. Christian scholarship is called to avoid such absolutisations.

the uniqueness of *multiplicity* and *wholeness* (*number* and *space*) while accounting for their *mutual coherence*.

List of references

AGAZZI, E. 1991. The problem of reductionism in science. Episteme: a series in the foundational, methodological, philosophical, psychological, sociological, and political aspects of the sciences, pure and applied. Volume 18. Edited by M. Bunge. McGill University*:* Foundations and Philosophy of Science Unit. Boston: Kluwer Academic Publishers.

ARISTOTLE. 2001. The basic works of Aristotle. Edited by Richard McKeon with an introduction by C.D.C. Reeve. (Originally published by Random House in 1941.) New York: The Modern Library.

BERNAYS, P. 1976. Abhandlungen zur Philosophie der Mathematik. Darmstadt: Wissenschaftliche Buchgesellschaft.

BOYER, C.B. 1959. The history of the calculus and its conceptual development. New York: Dover.

CANTOR, G. 1962 [1932]. Gesammelte Abhandlungen Mathematischen und Philosophischen Inhalts. Hildesheim: Olms.

CASSIRER, E. 1953. Substance and function. New York: Dover. (First edition of the English translation of Substanzbegriff und Funktionsbegriff: 1923.)

CASSIRER, E. 1957. Das Erkenntnisproblem in der Philosophie und Wissenschaft der neueren Zeit. Stuttgart: Kohlhammer.

DANTZIG, T. 1947. Number the language of science. London: Allen & Unwin.

DEDEKIND, R. 1969 [1887]. Was sind und was sollen die Zahlen. (10. Ausg.) Braunschweig: Friedrich Vieweg.

DUMMETT, M.A.E. 1995. Frege, philosophy of mathematics*.* Cambridge: Harvard University Press.

FELGNER, U., *Red.* 1979. Mengenlehre. Darmstadt: Wissenschaftliche Buchgesellschaft.

FRÄNKEL, H. 1968. Zeno von Elea im Kampf gegen die Idee der Vielheit. (*In* Gadamer, H-G., *Red.* Um die Begriffswelt der Vorsokratiker, Wege der Forschung. Band IX. Darmstadt: Wissenschaftliche Buchgesellschaft. p. 423-475.)

FRAENKEL, A., BAR-HILLEL, Y., LEVY, A. & VAN DALEN, D. 1973. Foundations of set theory. 2nd revised edition. Amsterdam: North Holland.

FREGE, G. 1979. Posthumous writings. Oxford: Blackwell.

FREUDENTHAL, H. 1940. Zur Geschichte der vollständigen Induktion. *Archives Internationales d'Histoire des Science*, 22:17-37.

GUTHRIE, W.K.C. 1980. A history of Greek philosophy. Volume II: The Presocratic tradition from Parmenides to Democritus. Cambridge: Cambridge University Press.

HASSE, H. & SCHOLZ, H. 1928. Die Grundlagenkrisis der griechischen Mathematic. *Kant-Studien*, 33:4-34.

HILBERT, D. 1922 [1899]. Grundlagen der Geometrie. 5. Ausg. Leipzig: Teubner.

HILBERT, D. 1925. Über das Unendliche. *Mathematische Annalen*, 95:161-190.

HILBERT, D. 1970 [1930]. Gesammelte Abhandlungen. Vol. III. 2.Ausg. Berlin: Springer.

HOYNINGEN-HUENE, P. 1991. Theory of antireductionist arguments: the Bohr case study. (*In* Agazzi, E. The problem of reductionism in science. Episteme: a series in the foundational, methodological, philosophical, psychological, sociological, and political aspects of the sciences, pure and applied. Volume 18. Edited by M. Bunge. McGill University*:* Foundations and Philosophy of Science Unit. Boston: Kluwer Academic Publishers. p. 51-70.)

KANT, I. 1967 [1787]. Kritik der reinen Vernunft. Hamburg: Felix Meiner.

MESCHKOWSKI, H. 1967. Problemen des Unendlichen. Braunschweig: Friedrich Vieweg.

MONK, J.D. 1970. On the foundations of set theory. *The American Mathematical Monthly*, 77:703-711.

MYHILL, J, 1952. Some philosophical implications of mathematical logic. *The Review of Metaphysics*, 6(2):165-198, Dec.

RUCKER, R. 1982. Infinity and the mind: the science and philosophy of the infinite. Boston: Birkhäuser.

RUSSELL, B. 1956. The principles of mathematics. London: Allen & Unwin. (First published in 1903, second edition 1937, seventh impression 1956.)

SALMON, W., *ed*. 2001. Zeno's paradoxes. New York: Bobs-Merrill.

SINGH, D. 1985. On Cantor's concept of set. *International Logical Review*, 32:72-85.

SKOLEM, Th. 1979 [1929]. Über die Grundlagendiskussionen in der Mathematik. (*In* Felgner, U., *Red.* Mengenlehre. Darmstadt: Wissenschaftliche Buchgesellschaft. p. 73-91.)

STRAUSS, D.F.M. 1991. The ontological status of the principle of the excluded middle. *Philosophia Mathematica*, II 6(1):73-90.

STRAUSS, D.F.M. 2002. Philosophical reflections on continuity. *Acta Academica*, 34(3):1-32.

VAN HEIJENOORT, J. 1967. From Frege to Gödel – a source book in mathematical logic 1879-1931. Cambridge: Harvard University Press.

VON FRITZ, K. 1945. The discovery of incommensurability by Hippasus of Metapontum. *Annals of Mathematics*, 46:242-264.

WANG, H. 1988. Reflections on Gödel. Cambridge: MIT.

WEINGARTNER, P. 1991. Reductionism and reduction in logic and in mathematics. (*In* Agazzi, E. The problem of reductionism in science. Episteme: a series in the foundational, methodological, philosophical, psychological, sociological, and political aspects of the sciences, pure and applied. Volume 18. Edited by M. Bunge. McGill University*:* Foundations and Philosophy of Science Unit. Boston: Kluwer Academic Publishers. p. 119-148.)

WEYL, H. 1921. Ueber die neue Grundlagenkrise der Mathematik. *Mathematische Zeitschrift*, 10:39-79.

WEYL, H. 1966. Philosophie der Mathematik und Naturwissenschaft. 3. Ausg. Vienna: Oldenburg.

Key concepts:

arithmeticism
geometricism
mutual coherence
primitive/basic terms
uniqueness
whole-parts relation

Kernbegrippe:

aritmetisisme
geheel-dele-verhouding
geometrisisme
primitiewe/basiese terme
uniekheid
wederkerige samehang